Lecture 10: September 30

Renormalizing the pairing. Last time, we computed a formula for the polarization $h_{\mathscr{V}}$, in the trivialization of $\mathscr{V}$ induced by the canonical extension. Recall that the canonical extension admits a (unique) trivialization $\tilde{\mathscr{V}} \cong \mathscr{O}_\Delta \otimes_{\mathbb{C}} V$ such that the logarithmic connection takes the simple form

$$\nabla(1 \otimes v) = \frac{dt}{t} \otimes Rv,$$

where $R \in \operatorname{End}(V)$ is the residue of the connection at the origin; the eigenvalues of R are real, and are contained in a fixed half-open interval $I \subseteq \mathbb{R}$ of length 1. Restricting to Δ^*, we get a trivialization of $\mathscr{V}$, and we showed that the polarization $h_{\mathscr{V}}$ of the variation of Hodge structure takes the form

$$h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') = \sum_{\alpha \in I} |t|^{2\alpha} \sum_{\ell=0}^{\infty} \frac{L(t)^\ell}{\ell!} (-1)^\ell h\big(v'_\alpha, R_N^\ell v''_\alpha\big).$$

Here $R = R_S + R_N$ is the Jordan decomposition of R, and $v'_\alpha, v_{\alpha''} \in E_\alpha(R_S)$ are the components of v', v'' in the α-eigenspace of R_S.

Our goal is to "renormalize" the pairing, in order to make the factors $|t|^{2\alpha} L(t)^\ell$ disappear. For that, we need a splitting of the monodromy weight filtration $W_\bullet = W_\bullet(R_N)$, so that we have well-defined subspaces on which we can do the rescaling. If you worked out the exercises from last time, you know that it is possible to choose a semisimple operator $H \in \operatorname{End}(V)$, with integer eigenvalues, and the following three properties:

(a) For every $j \in \mathbb{Z}$, one has $W_j = E_j(H) \oplus W_{j-1}$.
(b) One has $[H, R_N] = -2R_N$ and $[H, R_S] = 0$.
(c) One has $h(Hv', v'') = -h(v', Hv'')$ for every $v', v'' \in V$.

Since R_S and H are commuting semisimple operators, they have a simultaneous eigenspace decomposition

$$V = \bigoplus_{\substack{\alpha \in I \\ j \in \mathbb{Z}}} V_{\alpha,j}.$$

In this decomposition, $V_{\alpha,j}$ and $V_{\beta,k}$ are orthogonal with respect to the pairing h, except when $\alpha = \beta$ and $j + k = 0$; this follows from the identities $h(R_S v', v'') = h(v', R_S v'')$ and $h(Hv', v'') = -h(v', Hv'')$.

Now let us write $v' = \sum_{\alpha,j} v'_{\alpha,j}$, and similarly for v''. Then

$$\begin{aligned} h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') &= \sum_{\alpha \in I} |t|^{2\alpha} \sum_{\ell=0}^{\infty} \frac{L(t)^\ell}{\ell!} (-1)^\ell h\big(v'_\alpha, R_N^\ell v''_\alpha\big) \\ &= \sum_{\alpha \in I} |t|^{2\alpha} \sum_{\ell=0}^{\infty} \frac{L(t)^\ell}{\ell!} \sum_{j=0}^{\infty} (-1)^\ell h\big(v'_{\alpha,j}, R_N^\ell v''_{\alpha,2\ell-j}\big), \end{aligned}$$

because $v'_{\alpha,j}$ and $R_N^\ell v''_{\alpha,k}$ are h-orthogonal unless $j + k = 2\ell$. The formula suggests that we should rescale by the factor $|t|^{-\alpha} L(t)^{-j/2}$ on the subspace $V_{\alpha,j}$, in order to get rid of the terms involving t. Define two new vectors $w', w'' \in V$ by the rule

$$v'_{\alpha,j} = |t|^{-\alpha} L(t)^{-j/2} w'_{\alpha,j} \quad \text{and} \quad v''_{\alpha,j} = |t|^{-\alpha} L(t)^{-j/2} w''_{\alpha,j}.$$

Substituting into the formula from above gives

$$h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') = \sum_{\alpha \in I} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j=0}^{\infty} (-1)^\ell h\big(w'_{\alpha,j}, R_N^\ell w''_{\alpha,2\ell-j}\big) = h(w', e^{-R_N} w''),$$

and so all the divergent terms $|t|^{2\alpha} L(t)^{\ell}$ have indeed gone away. It remains to write the result in a more useful form. Since R_S acts on $V_{\alpha,j}$ as multiplication by α, and H as multiplication by j, it is easy to see that

$$w'_{\alpha,j} = |t|^{\alpha} L(t)^{j/2} v'_{\alpha,j} = e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2}\log L(t) H} v'_{\alpha,j},$$

and hence that $w' = e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2}\log L(t) H} v'$, with a similar formula for w''. Putting this back into the result of the above calculation, we get

$$\begin{aligned} h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'') &= h(w', e^{-R_N} w'') = h(e^{-\frac{1}{2} R_N} w', e^{-\frac{1}{2} R_N} w'') \\ &= h\Big(e^{-\frac{1}{2} R_N} e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2}\log L(t) H} v', e^{-\frac{1}{2} R_N} e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2}\log L(t) H} v''\Big). \end{aligned} \tag{10.1}$$

Convergence of the Hodge filtration. Now let us interpret the result. Recall that the Hodge bundles $F^p\mathscr{V}$, in the given trivialization of $\mathscr{V}$, are described by a holomorphic mapping $\Psi\colon \Delta^* \to \check{D}$. Since the variation of Hodge structure is polarized, this means that at each point $t \in \Delta^*$, the filtration $F_{\Psi(t)}$ gives a Hodge structure of weight n on the vector space V, which is polarized by the pairing $(v', v'') \mapsto h_{\mathscr{V}}(1 \otimes v', 1 \otimes v'')$. Because of (10.1), this means that the new filtration

$$e^{-\frac{1}{2} R_N} e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2}\log L(t) H} F_{\Psi(t)}$$

gives a Hodge structure of weight n on V that is polarized by the *constant* pairing h. But Hodge structures of this kind are exactly parametrized by the period domain, and so we conclude that

$$t \mapsto e^{-\frac{1}{2} R_N} e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2}\log L(t) H} \Psi(t)$$

is a (now only real-analytic) mapping from the punctured disk Δ^* into the period domain D. The discussion about renormalizing the pairing now suggests the following theorem, which is the central result in the theory of polarized variations of Hodge structure on the punctured disk.

Theorem 10.2. *With notation as above, the mapping*

$$\Delta^* \to D, \quad t \mapsto e^{-\frac{1}{2} R_N} e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2}\log L(t) H} \Psi(t),$$

extends continuously over the origin in Δ. More precisely, the following is true:

(a) *The limit*

$$\hat{F}_H = \lim_{t \to 0} e^{-\frac{1}{2} L(t) R_S} e^{\frac{1}{2}\log L(t) H} \Psi(t) \in \check{D}$$

exists, and $e^{-\frac{1}{2} R_N} \hat{F}_H \in D$. In other words, $e^{-\frac{1}{2} R_N} \hat{F}_H$ defines a Hodge structure of weight n on the vector space V, polarized by the pairing h.

(b) *The filtration $\hat{F}_H$ is compatible with the semisimple operators R_S and H, in the sense that $R_S \hat{F}_H^p \subseteq \hat{F}_H^p$ and $H \hat{F}_H^p \subseteq \hat{F}_H^p$ for every $p \in \mathbb{Z}$.*

(c) *One has $R_N \hat{F}_H^p \subseteq \hat{F}_H^{p-1}$ for every $p \in \mathbb{Z}$.*

This result contains parts of what Schmid calls the "nilpotent orbit theorem" and the "SL(2)-orbit theorem" in his paper. Schmid obtains a similar result near the end of his paper, after proving the two orbit theorems by another method; of course, his theorem is stated only for variations of Hodge structure that are defined over $\mathbb{R}$ and have quasi-unipotent monodromy.

Note. Let me stress again that this result is, at least conceptually, very simple: If we use the canonical extension to trivialize the vector bundle $\mathscr{V}$, and if we rescale the pairing to account for the different rates of growth $|t|^{2\alpha} L(t)^{\ell}$, then our family of polarized Hodge structures of weight n converges to a well-defined limit, which is again a polarized Hodge structure of weight n. (Of course, the limit depends on the choice of semisimple operator H.) People usually say that the limit is a "mixed Hodge structure", but I think our point of view makes more sense.

From Theorem 10.2, one can deduce most of the other results that Schmid proves in his paper (with the exception of the precise asymptotics in the SL(2)-orbit theorem), such as the famous estimates for the Hodge norm. We will also see how to deduce the existence of a "limiting mixed Hodge structure" (which Schmid deduces from his SL(2)-orbit theorem). This kind of result is best stated in the language of polarized Hodge-Lefschetz structures from Lecture 3, which now make another surprising appearance. Recall that Hodge-Lefschetz structures are representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ with compatible Hodge structures. In the case at hand, the vector space V becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$ by letting the matrix $H \in \mathfrak{sl}_2(\mathbb{C})$ act as the semisimple operator $H \in \operatorname{End}(V)$, and by letting $Y \in \mathfrak{sl}_2(\mathbb{C})$ act as the nilpotent operator $R_N \in \operatorname{End}(V)$.

Theorem 10.3. *Each eigenspace $V_k = E_k(H)$ has a Hodge structure of weight $n+k$, whose Hodge filtration agrees with the filtration induced by $\hat{F}_H$. With these Hodge structures, and with the $\mathfrak{sl}_2(\mathbb{C})$-action induced by H and R_N, the vector space*

$$V = \bigoplus_{k \in \mathbb{Z}} V_k$$

becomes a Hodge-Lefschetz structure of central weight n, polarized by the hermitian pairing h. Moreover, the operator $R_S \in \operatorname{End}(V)$ is an endomorphism of this Hodge-Lefschetz structure.

In particular, this says that the hermitian pairing

$$(v', v'') \mapsto (-1)^{\ell} h(v', R_N^{\ell} v'')$$

polarizes the Hodge structure of weight $n + \ell$ on the "primitive" subspace

$$\ker R_N^{\ell+1} \colon V_{\ell} \to V_{-\ell-2}.$$

You may remember that this expression already showed up as the coefficient of $L(t)^{\ell}$ in our formula for the pairing in Lecture 9. Once we have proved Theorem 10.2, we will see that Theorem 10.3 can be deduced by (very clever) linear algebra methods.

Some examples. Before diving into the proof of Theorem 9.1, let us go through a few concrete examples. In these examples, one can verify the general results from above by hand. Most of them are about Hodge structures of "elliptic curve" type, but things will look simpler if we use a different description than in Example 5.4.

Example 10.4. As in Example 5.4, we consider Hodge structures of weight 1 on $\mathbb{C}^2$, but this time, we write the hermitian pairing in the form

$$h\big((x', y'), (x'', y'')\big) = y'\overline{x''} + x'\overline{y''}.$$

This also has signature $(1,1)$, but differs from the pairing in Example 5.4 by a simple coordinate change. As before, the filtration is determined by the one-dimensional subspace $F^1 \subseteq \mathbb{C}$, and so $\check{D} = \mathbb{P}^1$. In order for the subspace

$$F^1 = \mathbb{C} \cdot (x, y)$$

to correspond to a polarized Hodge structure of weight 1, we need

$$0 < (-1)^1 h\big((x,y),(x,y)\big) = -\big(y\overline{x} + x\overline{y}\big)$$

or equivalently, $\operatorname{Re}(y\overline{x}) < 0$. Thus the period domain is $D = \tilde{\mathbb{H}}$ in this case, with a point $z \in \tilde{\mathbb{H}}$ corresponding to the Hodge structure

$$\mathbb{C} \cdot (1, z) \oplus \mathbb{C} \cdot (\overline{z}, 1).$$

Note that I am using $(1, z)$ instead of $(z, 1)$, in order to make the results in the next few examples come out more nicely. You can check that a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

lies in the subgroup $G_{\mathbb{R}}$ iff $a\bar{c}$ and $b\bar{d}$ are purely imaginary and $a\bar{d} + b\bar{c} = 1$. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} a + bt \\ c + dt \end{pmatrix},$$

such a matrix then acts on $\tilde{\mathbb{H}}$ by the formula $z \mapsto (a + bz)^{-1}(c + dz)$. (Again, we could have used the usual fractional transformations here, but then some of the results in the following examples would look less nice.)

Example 10.5. Given a polarized variation of Hodge structure of the above type on Δ^*, the period mapping is now simply a holomorphic mapping $f\colon \tilde{\mathbb{H}} \to \tilde{\mathbb{H}}$. If

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{R}}$$

is the monodromy matrix, then we have

$$f(z + 2\pi i) = \frac{c + dz}{a + bz}$$

for every $z \in \tilde{\mathbb{H}}$. Let us consider the special case where the monodromy is trivial, so $T = \mathrm{id}$. Here Theorem 10.2 (with $R = H = 0$) is claiming that the period mapping extends holomorphically to the entire disk. We can verify this as follows. From $f(z + 2\pi i) = f(z)$, we get than $f(z) = g(e^z)$ for a holomorphic function $g\colon \Delta^* \to \tilde{\mathbb{H}}$. Now $\tilde{\mathbb{H}}$ is isomorphic to the unit disk, and so the Riemann extension theorem implies that g extends to a holomorphic function $g\colon \Delta \to \mathbb{C}$. Since g is an open mapping, it follows that $g(0) \in \tilde{\mathbb{H}}$; now $g\colon \Delta \to \tilde{\mathbb{H}}$ is the desired extension of the period mapping.

Example 10.6. For another example of a period mapping, we can take

$$f\colon \tilde{\mathbb{H}} \to \tilde{\mathbb{H}}, \quad f(z) = z.$$

In this case, the monodromy matrix can be

$$T = \begin{pmatrix} 1 & 0 \\ 2\pi i & 1 \end{pmatrix},$$

and if we work with the interval $I = [0, 1)$, then

$$R = R_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is one of the standard generators of $\mathfrak{sl}_2(\mathbb{C})$. According to Theorem 9.1, we should consider the "untwisted period mapping"

$$e^{-zR} f(z) = \begin{pmatrix} 1 & 0 \\ -z & 0 \end{pmatrix} \cdot z = 0.$$

Note that the limit trivially exists, but is *not* a point of the period domain $\tilde{\mathbb{H}}$. Indeed, $\Psi(0)$ corresponds to the filtration with

$$F^1 = \mathbb{C} \cdot (1, 0),$$

but the pairing h is *not* negative definite on this subspace.

Example 10.7. Lastly, let us consider the special case of unipotent monodromy. Suppose that $f\colon \tilde{\mathbb{H}} \to \tilde{\mathbb{H}}$ is a holomorphic mapping with

$$f(z + 2\pi i) = f(z) + 2\pi i c,$$

which corresponds to taking

$$T = \begin{pmatrix} 1 & 0 \\ 2\pi i c & 1 \end{pmatrix}.$$

The fact that $T \in G_{\mathbb{R}}$ means that $c \in \mathbb{R}$ is real; for simplicity, let us suppose that $c \in \mathbb{Z}$. Once again,

$$e^{-zR} = \begin{pmatrix} 1 & 0 \\ -cz & 1 \end{pmatrix},$$

and so the "untwisted period mapping" corresponds to the function $f(z) - cz$, which is invariant under the substitution $z \mapsto z + 2\pi i$. This gives us

$$f(z) = cz + g(e^z),$$

for a holomorphic function $g\colon \Delta^* \to \mathbb{C}$. Now Theorem 9.1 is saying that g extends to a holomorphic function on the entire disk Δ. In this toy example, we can prove this as follows. Exponentiating both sides and writing $t = e^z$, we get

$$t^c e^{g(t)} = e^{f(z)} \in \Delta^*,$$

and so by Riemann's extension theorem, $t^c e^{g(t)}$ extends holomorphically to Δ. Near the origin, we can write the resulting holomorphic function as $t^k e^{h(t)}$, where $k \geq 0$ is the order of vanishing at the origin, and $h(t)$ is holomorphic. Then

$$e^{g(t)-h(t)} = t^{k-c},$$

and if $k - c \neq 0$, then $\frac{g(t)-h(t)}{k-c}$ would be a holomorphic logarithm function on a punctured neighborhood of the origin, which we know cannot exist. So the conclusion is that $e^{g(t)}$, and hence also $g(t)$ itself, extends holomorphically to the entire disk; we also find that $c \geq 0$.

Exercise 10.1. In the setting of Theorem 10.2, there is another way to see why

$$e^{-\frac{1}{2}R_N} e^{-\frac{1}{2}L(t)R_S} e^{\frac{1}{2}\log L(t) H} \Psi(t) \in D.$$

Here is how this goes:

(a) Prove the identity

$$e^{-\frac{1}{2}R_N} e^{-\frac{1}{2}L(t)R_S} e^{\frac{1}{2}\log L(t) H} e^{-zR} = e^{\frac{1}{2}\log L(t) H} e^{\frac{1}{2}(\bar{z}-z)R}.$$

(Hint: First check what happens on the subspace $V_{\alpha,j}$.)

(b) Show that the Lie algebra of the group $G_{\mathbb{R}}$ consists of all $A \in \operatorname{End}(V)$ with the property that $A^* = -A$, where A^* is the adjoint with respect to h.

(c) Conclude that if $x \in \mathbb{R}$ is real, and $y \in \mathbb{C}$ purely imaginary, then both e^{xH} and e^{yR} are elements of $G_{\mathbb{R}}$.